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MODAL MASS, STIFFNESS AND DAMPING

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INTRODUCTION

For classically damped structures, modal mass, stiffness and damping can be defined directly from formulas that relate the full mass, stiffness and damping matrices to the transfer function matrix. The modal mass, stiffness, and damping definitions are derived in a previous paper [1], and are re-stated here for convenience.

The transfer function is defined over the complex Laplace plane, as a function of the variable $(\mathbf{s} = \sigma + \mathbf{j}\omega)$. Experimentally, the values of a transfer function are measured only along the $\mathbf{j}\omega$-axis in the **s-plane**, that is for $(\mathbf{s} = \mathbf{j}\omega)$. These values are referred to as the **Frequency Response Function** (**FRF**).

CLASSICALLY DAMPED STRUCTURE

A classically damped structure is one where the modal damping is ***much smaller*** than the damped natural frequency of each mode (it is lightly damped), and the mode shapes are ***primarily real valued*** (they approximate normal modes).

Light Damping: A structure is lightly damped if the damping coefficient $(\sigma_{\mathbf{k}})$ of each mode (**k**) is much less than the damped natural frequency $(\omega_{\mathbf{k}})$. That is,

$$\sigma_{\mathbf{k}} << \omega_{\mathbf{k}} \qquad (1)$$

Normal Mode Shapes: If the imaginary part of each mode shape vector $\{\mathbf{u}_\mathbf{k}\}$ is much less than the real part, that is if,

$$\mathbf{Im}\left(\{\mathbf{u}_\mathbf{k}\}\right) << \mathbf{Re}\left(\{\mathbf{u}_\mathbf{k}\}\right) \qquad (2)$$

where,

$$\{\mathbf{u}_\mathbf{k}\} = \mathbf{Re}\left(\{\mathbf{u}_\mathbf{k}\}\right) + \mathbf{j}\,\mathbf{Im}\left(\{\mathbf{u}_\mathbf{k}\}\right) \qquad (3)$$

the structure's mode shapes ***approximate normal modes***.

Both of these assumptions are satisfied by a large variety of real structures from which experimental modal data has been acquired.

MODAL MASS MATRIX

When the mass matrix is post-multiplied by the mode shape matrix and pre-multiplied by its transpose, the result is a diagonal matrix, shown in equation (4). ***This is a definition of modal mass.***

$$[\phi]^t[\mathbf{M}][\phi] = \left[\ddots\, \mathbf{m} \,\ddots\right] = \begin{bmatrix} \ddots & & \\ & \dfrac{\mathbf{1}}{\mathbf{A}\omega} & \\ & & \ddots \end{bmatrix} \qquad (4)$$

where,

$[\mathbf{M}]$ = **(n by n)** mass matrix.

$[\phi] = \left[\{\mathbf{u}_1\}\{\mathbf{u}_2\}\ldots\{\mathbf{u}_\mathbf{m}\}\right]$ = **(n by m)** mode shape matrix.

$\{\mathbf{u}_\mathbf{k}\}$ = **n**-dimensional mode shape vector for the k^{th} mode, **k = 1 to m**.

m = number of modes of vibration.

n = number of DOFs of the structure model.

$\left[\ddots\, \mathbf{m} \,\ddots\right] = \begin{bmatrix} \ddots & & \\ & \dfrac{\mathbf{1}}{\mathbf{A}\omega} & \\ & & \ddots \end{bmatrix}$ = (**m by m**) modal mass matrix.

The modal mass of each mode (**k**) is a diagonal element of the modal mass matrix,

Modal mass: $$\mathbf{m}_\mathbf{k} = \frac{\mathbf{1}}{\mathbf{A}_\mathbf{k}\omega_\mathbf{k}} \qquad \mathbf{k = 1\ to\ m} \qquad (5)$$

$\mathbf{p}_\mathbf{k} = -\sigma_\mathbf{k} + \mathbf{j}\omega_\mathbf{k}$ = pole location for the k^{th} mode.

$\sigma_\mathbf{k}$ = damping coefficient of the k^{th} mode.

$\omega_\mathbf{k}$ = damped natural frequency of the k^{th} mode.

$\mathbf{A}_\mathbf{k}$ = a scaling constant for the k^{th} mode.

MODAL STIFFNESS MATRIX

When the stiffness matrix is post-multiplied by the mode shape matrix and pre-multiplied by its transpose, the result is a diagonal matrix, shown in equation (6). ***This is a definition of modal stiffness.***

$$[\phi]^t[\mathbf{K}][\phi] = \left[\ddots\, \mathbf{k} \,\ddots\right] = \begin{bmatrix} \ddots & & \\ & \dfrac{\sigma^2 + \omega^2}{\mathbf{A}\omega} & \\ & & \ddots \end{bmatrix} \qquad (6)$$

where,

$[\mathbf{K}]$ = **(n by n)** stiffness matrix.

$\left[\ddots \mathbf{k} \ddots\right] = \begin{bmatrix} \ddots & & \\ & \dfrac{\sigma^2 + \omega^2}{\mathbf{A}\omega} & \\ & & \ddots \end{bmatrix}$ = (**m by m**) modal stiffness matrix.

The modal stiffness of each mode (**k**) is a diagonal element of the modal stiffness matrix,

$$\textbf{Modal stiffness: } \mathbf{k_k} = \frac{\sigma_k^2 + \omega_k^2}{\mathbf{A_k}\omega_k} \qquad \mathbf{k = 1 \text{ to } m} \qquad (7)$$

MODAL DAMPING MATRIX

When the damping matrix is post-multiplied by the mode shape matrix and pre-multiplied by its transpose, the result is a diagonal matrix, shown in equation (8). ***This is a definition of modal damping.***

$$[\phi]^t[\mathbf{C}][\phi] = \left[\ddots \mathbf{c} \ddots\right] = \begin{bmatrix} \ddots & & \\ & \dfrac{2\sigma}{\mathbf{A}\omega} & \\ & & \ddots \end{bmatrix} \qquad (8)$$

where,

$[\mathbf{C}]$ = **(n by n)** damping matrix.

$\left[\ddots \mathbf{c} \ddots\right] = \begin{bmatrix} \ddots & & \\ & \dfrac{2\sigma}{\mathbf{A}\omega} & \\ & & \ddots \end{bmatrix}$ = (**m by m**) modal damping matrix.

The modal damping of each mode (**k**) is a diagonal element of the modal damping matrix,

$$\textbf{Modal damping: } \mathbf{c_k} = \frac{2\sigma_k}{\mathbf{A_k}\omega_k} \qquad \mathbf{k = 1 \text{ to } m} \qquad (9)$$

SDOF RELATIONSHIPS

The familiar single degree-of-freedom (SDOF) relationships follow from the definitions of modal mass, stiffness, and damping for multiple DOF systems,

$$\frac{\mathbf{k_k}}{\mathbf{m_k}} = (\sigma_k^2 + \omega_k^2) \qquad \mathbf{k = 1 \text{ to } m} \qquad (10)$$

$$\frac{\mathbf{c_k}}{\mathbf{m_k}} = (2\sigma_k) \qquad \mathbf{k = 1 \text{ to } m} \qquad (11)$$

SCALING MODE SHAPES TO UNIT MODAL MASSES

Mode shapes are called "***shapes***" because they are unique in shape, but not in value. That is, the mode shape vector $\{\mathbf{u_k}\}$ for each mode (**k**) does not have unique values. It can be arbitrarily scaled to any set of values, but the relationship of one shape component to any other is unique. In other words, the "***shape***" of $\{\mathbf{u_k}\}$ is unique, but its values are not. A mode shape is also called an ***eigenvector***, which means that its "***shape***" is unique, but its values are arbitrary.

Notice also, that each of the modal mass, stiffness, and damping matrix definitions (5), (7), and (9) includes a ***scaling constant*** $(\mathbf{A_k})$. This constant is necessary because the mode shapes are not unique in value, and therefore can be arbitrarily scaled.

Unit Modal Masses

One of the common ways to scale mode shapes is to scale them so that the modal masses are one (unity). Normally, if the mass matrix $[\mathbf{M}]$ were available, the mode vectors would simply be scaled such that when the triple product $[\mathbf{U}]^t[\mathbf{M}][\mathbf{U}]$ was formed, the resulting modal mass matrix would equal an ***identity matrix***. However, when the modal data is obtained from experimental transfer function measurements (FRFs), no mass matrix is available for scaling in this way.

Even without the mass matrix however, experimental mode shapes can still be scaled to unit modal masses by using the relationship between residues and mode shapes.

$$[\mathbf{r(k)}] = \mathbf{A_k}\{\mathbf{u_k}\}\{\mathbf{u_k}\}^t \qquad \mathbf{k = 1 \text{ to } m} \qquad (12)$$

where,

$[\mathbf{r(k)}]$ = (**n by n**) residue matrix for the k^{th} mode.

Residues are the constant numerators of the transfer function matrix when it is written in partial fraction form,

$$[\mathbf{H(s)}] = \sum_{k=1}^{m} \frac{[\mathbf{r(k)}]}{2\mathbf{j}(\mathbf{s} - \mathbf{p_k})} - \frac{[\mathbf{r(k)}]^*}{2\mathbf{j}(\mathbf{s} - \mathbf{p_k^*})} \qquad (13)$$

* -denotes the complex conjugate.

Residues have unique values, and have engineering units. Since the transfer functions typically have units of (motion / force), and the denominators have units of Hz or (radians/second), residues have units of (motion / force) (Hz).

Equation (12) can be written for the $\mathbf{j^{th}}$ column (or row) of the residue matrix and for mode **(k)** as,

$$\begin{Bmatrix} r_{1j}(k) \\ r_{2j}(k) \\ \vdots \\ \vdots \\ r_{jj}(k) \\ \vdots \\ r_{nj}(k) \end{Bmatrix} = A_k \begin{Bmatrix} u_{1k}u_{jk} \\ u_{2k}u_{jk} \\ \vdots \\ \vdots \\ (u_{jk})^2 \\ \vdots \\ u_{nk}u_{nk} \end{Bmatrix} = A_k u_{jk} \begin{Bmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ \vdots \\ u_{jk} \\ \vdots \\ u_{nk} \end{Bmatrix} \qquad (14)$$

Unique **Variable** **k=1,…, m**

The importance of this relationship is that ***residues are unique in value*** and reflect the unique physical properties of the structure, while the ***mode shapes aren't unique in value*** and can therefore be scaled in any manner desired.

The scaling constant A_k must always be chosen so that equation (14) remains valid. The value of A_k can be chosen first, and the mode shapes scaled accordingly so that equation (14) is satisfied. Or, the mode shapes can be scaled first and A_k computed so that equation (14) is still satisfied.

In order to obtain mode shapes scaled to unit modal masses, we simply set the modal mass to one (1) and solve equation (5) for A_k,

$$A_k = \frac{1}{w_k} \qquad \mathbf{k=1\ to\ m} \qquad (15)$$

Driving Point Measurement

The unit modal mass scaled mode shape vectors are obtained from the j^{th} column (or row) of the residue matrix by substituting equation (15) into equation (14),

$$\begin{Bmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ \vdots \\ \vdots \\ u_{nk} \end{Bmatrix} = \frac{1}{A_k u_{jk}} \begin{Bmatrix} r_{1j}(k) \\ r_{2j}(k) \\ \vdots \\ \vdots \\ \vdots \\ r_{nj}(k) \end{Bmatrix} = \sqrt{\frac{w_k}{r_{jj}(k)}} \begin{Bmatrix} r_{1j}(k) \\ r_{2j}(k) \\ \vdots \\ \vdots \\ \vdots \\ r_{nj}(k) \end{Bmatrix} \qquad (16)$$

UMM **k=1,…, m**

Notice that the ***driving point residue*** $r_{jj}(k)$ (where the row index(**j**) equals the column index(**j**)), plays an important role in this scaling process. Therefore, the driving point residue for each mode(**k**) is required in order to use equation (16).

Triangular Measurement

For cases where the driving point measurement cannot be made, an alternative set of measurements can be used to provide the driving point mode shape component u_{jk}. From equation (14) we can write,

$$u_{jk} = \sqrt{\frac{A_k\ r_{jp}(k)\, r_{jq}(k)}{r_{pq}(k)}} \qquad \mathbf{k=1\ to\ m} \qquad (17)$$

Equation (17) can be substituted for u_{jk} in equation (16) to yield mode shapes scaled to unit modal masses. Equation (17) says that as an alternative to making a driving point measurement, three other measurements can be made involving DOF(**p**), DOF(**q**), and DOF(**j**).

DOF(**j**) is the reference (fixed) DOF for the j^{th} column (or row) of transfer function measurements, so the two measurements H_{jp} and H_{jq} would normally be made. In addition, one extra measurement H_{pq} is also required in order to solve equation (17). Since the measurements H_{jp}, H_{jq}, and H_{pq} form a triangle in the transfer function matrix, they are called a ***triangular measurement***.

CONVERTING RESIDUES TO DISPLACEMENT UNITS

Vibration measurements are often made using accelerometers to measure acceleration response, or vibrometers to measure velocity. Excitation forces are typically measured with a load cell. Therefore, transfer function measurements made with these transducers will have units of either (**acceleration/force**) or (**velocity/force**).

Modal residues always carry the units of the transfer function multiplied by (**radians/second**). Therefore, residues taken from transfer functions with units of (acceleration/force) will have units of **(acceleration/force-sec).** Likewise**,** residues taken from measurements with units of (velocity/force) would have units of **(velocity/force-sec).** Similarly, residues taken from measurements with units of (displacement/force-sec) would have units of (**displacement/force-sec**).

Since the modal mass, stiffness, and damping equations (4), (6), and (8) ***assume units of*** **(displacement/force)**, residues with units of (acceleration/force-sec) or (velocity/force-sec) must be "integrated" to units of (displacement/force-sec) units before performing mode shape scaling.

Integration of a time domain function has an equivalent operation in the frequency domain. Integration of a transfer function is done by dividing it by the Laplace variable(**s**),

$$[H_d(s)] = \frac{[H_v(s)]}{s} = \frac{[H_a(s)]}{s^2} \qquad (18)$$

where,

$[H_d(s)]$ = transfer matrix in (**displacement**/force) units.

$[H_v(s)]$ = transfer matrix in (**velocity**/force) units.

$[H_a(s)]$ = transfer matrix in (**acceleration**/force) units.

Since residues are the result of a partial fraction expansion of a transfer function, residues can be "***integrated***" directly as if they were obtained from an integrated transfer function using the formula,

$$[r_d(k)] = \frac{[r_v(k)]}{p_k} = \frac{[r_a(k)]}{(p_k)^2} \qquad \textbf{k=1 to m} \qquad (19)$$

where,

$[r_d(k)]$ = residue matrix in (**displacement**/force) units.

$[r_v(k)]$ = residue matrix in (**velocity**/force) units.

$[r_a(k)]$ = residue matrix in (**acceleration**/force) units.

$p_k = -\sigma_k + j\omega_k$ = pole location for the k^{th} mode.

Since we are assuming that damping is light and the mode shapes are normal, equation (19) can be simplified to,

$$[r_d(k)] = \Phi_k\,[r_v(k)] = (\Phi_k)^2\,[r_a(k)] \qquad \textbf{k=1 to m} \qquad (20)$$

where,

$$\Phi_k = \frac{\omega_k}{(\sigma_k^2 + \omega_k^2)} \qquad \textbf{k=1 to m} \qquad (21)$$

Equations (20) and (21) can be summarized in the following table.

To change transfer function units		Multiple residues
From	**To**	**By**
ACCELERATION / FORCE	DISPLACEMENT / FORCE	Φ^2
VELOCITY / FORCE	DISPLACEMENT / FORCE	Φ

Table 1. Residue Scale Factors.

Where: $$\Phi = \frac{\omega}{(\sigma^2 + \omega^2)} \qquad \textbf{(seconds)}$$

EXAMPLE OF UNIT MODAL MASS SCALING

Suppose that we have the following data for a single mode of vibration,

Frequency = 10.0 Hz.

Damping = 1.0 %

$$\textbf{Residue Vector} = \begin{Bmatrix} -0.1 \\ +2.0 \\ +0.5 \end{Bmatrix}$$

Also, suppose that the measurements from which this data was obtained have units of (**Gs/Lbf)**. Also assume that the driving point is at the second DOF of the structure. Hence the ***driving point residue*** **= 2.0**.

Converting the frequency and damping into units of ***radians/second***,

Frequency = 62.83 Rad/Sec

Damping = 0.628 Rad/Sec

The residues always carry the units of the transfer function measurement multiplied by (**radians/second**). Therefore, for this case the units of the residues are,

Residue Units = Gs/(Lbf-Sec) = 386.4 Inches/(Lbf-Sec^3)

Therefore, the residues become,

$$\textbf{Residue Vector} = \begin{Bmatrix} -38.64 \\ +772.8 \\ +193.2 \end{Bmatrix} \ \textbf{Inches/(Lbf-Sec}^3\textbf{)}$$

Since the modal mass, stiffness, and damping equations (4), (6), and (8) assume units of (**displacement/force**), the above residues with units of (acceleration/force) have to be converted to (displacement/force) units. This is done by using the appropriate scale factor from Table 1. For this case:

$$\Phi^2 \cong \left(\frac{1}{62.83}\right)^2 = 0.000253 \quad (\text{Seconds}^2)$$

Multiplying the residues by Φ^2 gives,

$$\textbf{Residue Vector} = \begin{Bmatrix} -0.00977 \\ +0.1955 \\ +0.0488 \end{Bmatrix} \ \textbf{Inches/(Lbf-Sec)}$$

Finally, to obtain a mode shape scaled to unit modal mass, Equation (18) is used. The mode shape of residues must be multiplied by the scale factor,

$$\mathbf{SF} = \sqrt{\frac{\omega}{\mathbf{r_{jj}}}} = \sqrt{\frac{\mathbf{62.83}}{+\,\mathbf{0.1955}}} = \mathbf{17.927}$$

to obtain the unit modal mass mode shape,

$$\textbf{UMM Mode Shape} = \begin{Bmatrix} -0.175 \\ +3.505 \\ +0.875 \end{Bmatrix} \ \textbf{Inches/(Lbf-Sec)}$$

REFERENCES

[1] Richardson, M.H. "Derivation of Mass, Stiffness and Damping Parameters From Experimental Modal Data" Hewlett Packard Company, Santa Clara Division, June, 1977.

[2] Potter, R. and Richardson, M.H. "Mass, Stiffness and Damping Matrices from Measured Modal Parameters",.I.S.A. International Instrumentation - Automation Conference, New York, New York, October 1974